

## SEMIDUALIZING DG MODULES OVER TENSOR PRODUCTS

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**ABSTRACT.** We study the existence of nontrivial semidualizing DG modules over tensor products of DG algebras over a field. In particular, this gives a lower bound on the number of semidualizing DG modules over the tensor product.

## 1. INTRODUCTION

**Assumption 1.1.** Let  $R$  be a commutative, noetherian ring with identity.

Semidualizing modules were introduced by Foxby [9], while Vasconcelos [16] and Golod [12] rediscovered them independently and applied them in different contexts. A finitely generated  $R$ -module  $C$  is *semidualizing* over  $R$  if the homothety map  $\chi_C^R : R \rightarrow \operatorname{Hom}_R(C, C)$  is an isomorphism and  $\operatorname{Ext}_R^i(C, C) = 0$  for all  $i > 0$ . Let  $\mathfrak{S}_0(R)$  denote the set of isomorphism classes of semidualizing  $R$ -modules. The size of  $\mathfrak{S}_0(R)$  measures the severity of the singularity of a ring, specifically how close a ring is to being Gorenstein. If  $\mathfrak{S}_0(R)$  is large, then  $R$  is far from being Gorenstein. If  $\mathfrak{S}_0(R)$  is small, then  $R$  is in a sense close to being Gorenstein. For instance, if  $R$  is Gorenstein and local, then  $|\mathfrak{S}_0(R)| = 1$ .

Throughout this paper, we use the more general definition of semidualizing DG module. (“DG” is short for “Differential Graded”. See Section 2 for relevant background information.) The idea for the definition is essentially from Christensen and Sather-Wagstaff [8]; see also [13]. The DG setting has been useful for answering questions about rings. For instance, Nasseh and Sather-Wagstaff [13] were able to answer Vasconcelos’ question [16, p. 97], showing a local ring has only finitely many isomorphism classes of semidualizing modules.

**Definition 1.2.** Let  $A$  be a DG  $R$ -algebra. A *semidualizing* DG  $A$ -module is a homologically finite DG  $A$ -module  $C$  that admits a degreewise finite semifree resolution over  $A$  such that the homothety morphism  $\chi_C^A : A \rightarrow \mathbf{R}\operatorname{Hom}_A(C, C)$  is an isomorphism in the derived category  $D(A)$ . Let  $\mathfrak{S}(A)$  denote the set of shift-isomorphism classes of semidualizing DG  $A$ -modules in  $D(A)$ .

What follows is the main result of this paper, which is proven in 4.8. The big picture idea here is that the singularity of the ring  $A' \otimes_k A''$  is at least as bad as the singularities of both  $A'$  and  $A''$  combined.

**Theorem 1.3.** *Let  $k$  be a field. Let  $A'$  and  $A''$  be local DG  $k$ -algebras such that  $A'_0$  and  $A''_0$  are noetherian. Let  $M' \in D_b^f(A')$  and  $M'' \in D_b^f(A'')$ .*

- (a) *One has  $M' \otimes_k M''$  is semidualizing over  $A' \otimes_k A''$  if and only if  $M'$  is semidualizing over  $A'$  and  $M''$  is semidualizing over  $A''$ .*
- (b) *The map  $\psi : \mathfrak{S}(A') \times \mathfrak{S}(A'') \rightarrow \mathfrak{S}(A' \otimes_k A'')$  defined by  $\psi(C', C'') = C' \otimes_k C''$  is well-defined and injective.*

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Note that part (a) is a consequence of the Künneth Theorem, properly interpreted. However, parts (b) and (c) use an extension of Foxby and Christensen's Bass classes to the DG setting.

## 2. BACKGROUND

For a thorough introduction to DG algebras see any of the following [3, 4, 5, 6, 13]. Below is a quick review of a few of the necessary definitions.

**Remark 2.1.** In this paper  $R$ -complexes are indexed homologically, and  $|a| = i$  means  $a \in X_i$ .

**Definition 2.2.** A *commutative differential graded algebra over  $R$*  (“DG  $R$ -algebra” for short) is an  $R$ -complex  $A$  equipped with binary operations  $\mu^A : A_i \times A_j \rightarrow A_{i+j}$  with  $ab := \mu^A(a, b)$  satisfying the following properties:

- associative:** for all  $a, b, c \in A$  we have  $(ab)c = a(bc)$ ;
- distributive:** for all  $a, b, c \in A$  such that  $|a| = |b|$  we have  $(a+b)c = ac+bc$  and  $c(a+b) = ca+cb$ ;
- unital:** there is an element  $1 \in A_0$  such that for all  $a \in A$  we have  $1a = a$ ;
- graded commutative:** for all  $a, b \in A$  we have  $ab = (-1)^{|a||b|}ba$  and  $a^2 = 0$  when  $|a|$  is odd;
- positively graded:**  $A_i = 0$  for  $i < 0$ ; and
- Leibniz Rule:** for all  $a, b \in A$  we have  $\partial_{|a|+|b|}^A(ab) = \partial_{|a|}^A(a)b + (-1)^{|a|}a\partial_{|b|}^B(b)$ .

Given a DG  $R$ -algebra  $A$ , the *underlying algebra* is the graded commutative  $R$ -algebra  $A^\natural = \bigoplus_{i=0}^\infty A_i$ . We say that  $A$  is *weakly noetherian* if  $H_0(A)$  is noetherian and the  $H_0(A)$ -module  $H_i(A)$  is finitely generated for all  $i \geq 0$ . We say that  $A$  is *mildly noetherian* if  $A$  is weakly noetherian and  $A_0$  is noetherian. We say that  $A$  is *local* if it is weakly noetherian,  $R$  is local, and the ring  $H_0(A)$  is a local  $R$ -algebra. A *morphism* of DG  $R$ -algebras is a chain map  $f : A \rightarrow B$  between DG  $R$ -algebras respecting products and multiplicative identities:  $f(aa) = f(a)f(a)$  and  $f(1) = 1$ .

**Assumption 2.3.** For the rest of this section  $A$  is a DG  $R$ -algebra and  $k$  is a field.

**Definition 2.4.** A *differential graded module over  $A$*  (“DG  $A$ -module” for short) is an  $R$ -complex  $M$  equipped with binary operations  $\mu^M : A_i \times M_j \rightarrow M_{i+j}$  with  $am := \mu^M(a, m)$  satisfying the following properties:

- associative:** for all  $a, b \in A$  and  $m \in M$  we have  $(ab)m = a(bm)$
- distributive:** for all  $a, b \in A$  and  $m, n \in M$  such that  $|a| = |b|$  and  $|m| = |n|$ , we have  $(a+b)m = am+bm$  and  $a(m+n) = am+an$ ;
- unital:** for all  $m \in M$  we have  $1m = m$ ;
- Leibniz Rule:** for all  $a \in A$  and  $m \in M$  we have  $\partial_{|a|+|m|}(am) = \partial_{|a|}(a)m + (-1)^{|a|}a\partial_{|m|}(m)$ .

The *underlying  $A^\natural$ -module* associated to  $M$  is the  $A^\natural$ -module  $M^\natural = \bigoplus_{i=-\infty}^\infty M_i$ . Let  $D(A)$  denote the derived category of DG  $A$ -modules. Isomorphisms in  $D(A)$  are identified by the symbol  $\simeq$ .

**Definition 2.5.** A DG  $A$ -module  $M$  is *degreewise finite*, denoted  $M \in D^f(A)$ , if  $H_i(M)$  is finitely generated over  $H_0(A)$  for all  $i$ . We say  $M$  is *homologically bounded*, denoted  $M \in D_b(A)$ , if  $H_i(M) = 0$  for  $|i| \gg 0$ . Additionally,  $M$  is *homologically finite* if  $M \in D_b(A) \cap D^f(A)$ , i.e.,  $M \in D_b^f(A)$ . We say  $M$  is *homologically bounded below*, denoted  $M \in D_+(A)$ , if  $\inf(M) > -\infty$ . Similarly,  $M$  is *homologically bounded above*, denoted  $M \in D_-(A)$ , if  $\sup(M) < \infty$ .

**Fact 2.6.** Let  $A'$  and  $A''$  be DG  $R$ -algebras. Let  $N'$  be a DG  $A'$ -module and  $N''$  be a DG  $A''$ -module. Then

- (a)  $A' \otimes_R A''$  is a DG  $R$ -algebra via the multiplication  $(a' \otimes a'')(b' \otimes b'') = (-1)^{|a''||b'|}(a'b') \otimes (a''b'')$ , and
- (b) the complex  $N' \otimes_R N''$  is a DG  $A' \otimes_R A''$ -module via the multiplication  $(a' \otimes a'')(n' \otimes n'') = (-1)^{|a''||n'|}(a'n') \otimes (a''n'')$ .

**Definition 2.7.** A *semibasis* for a DG  $A$ -module  $M$  is a set  $E = \bigsqcup_{i=0}^{\infty} E^i$  such that  $\partial(E^i) \subseteq AE^{i-1}$  for each  $i \geq 0$  (we set  $AE^{-1} = 0$ ) and  $E$  is a basis of the  $A^\natural$ -module  $M^\natural$ . We say  $M$  is *semifree* if it has a semi-basis. A *semifree resolution* of a DG  $A$ -module  $N$  is a quasiisomorphism  $F \xrightarrow{\sim} N$  such that  $F$  is semi-free over  $A$ . We say that a DG  $A$ -module  $M$  is *semiprojective* if  $\mathrm{Hom}_A(M, -)$  respects surjective quasiisomorphisms. A *semiprojective resolution* of a DG  $A$ -module  $N$  is a quasiisomorphism  $P \xrightarrow{\sim} N$  such that  $P$  is semiprojective over  $A$ .

**Fact 2.8.** Let  $M$  and  $N$  be DG  $A$ -modules.

- (a) There exists a semifree resolution  $F \xrightarrow{\sim} M$ .
- (b) If  $A$  is weakly noetherian and  $M \in D_+^f(A)$ , then there exists a semifree resolution  $F \xrightarrow{\sim} M$  with semibasis  $E$  such that  $|E \cap M_n| < \infty$ . We call such a resolution a “degree-wise finite semifree resolution.”
- (c) If  $F$  is semifree over  $A$ , then  $F$  is semiprojective over  $A$ .
- (d) If  $M \in D_b(A)$ , then  $M$  is semifree over  $A$  if and only if  $M^\natural$  is a free graded  $A^\natural$ -module.

The following notion was defined for dualizing modules by Foxby [10] and for an arbitrary semidualizing module or complex by Christensen [7].

**Definition 2.9.** Let  $C \in \mathfrak{S}(A)$  and  $M \in D_b(A)$ . Then  $M$  is in the *Bass class*  $\mathcal{B}_C(A)$  if the natural evaluation morphism  $\xi_M^C : C \otimes_A^L \mathbf{R}\mathrm{Hom}_A(C, M) \rightarrow M$  is an isomorphism in  $D(A)$  and  $\mathbf{R}\mathrm{Hom}_A(C, M) \in D_b(A)$ .

Notice, if we are working over  $R$ , with  $C \in \mathfrak{S}_0(R)$  and  $M$  an  $R$ -module, this translates as follows:  $M$  is in the Bass class  $\mathcal{B}_C(R)$  if the natural evaluation homomorphism  $\xi : C \otimes_R \mathrm{Hom}_R(C, M)$  is an isomorphism and  $\mathrm{Ext}_R^i(C, M) = 0 = \mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, M))$  for all  $i > 0$ .

Similarly, we have the following notions of the Auslander class and derived reflexive DG modules.

**Definition 2.10.** Let  $C \in \mathfrak{S}(A)$  and  $M \in D_b(A)$ . Then  $M$  is in the *Auslander class*  $\mathcal{A}_C(A)$  if the natural morphism  $\gamma_M^C : M \rightarrow \mathbf{R}\mathrm{Hom}_A(C, C \otimes_A M)$  is an isomorphism in  $D(A)$  and  $C \otimes_A^L M \in D_b(A)$ .

If we are working over  $R$ , with  $C \in \mathfrak{S}_0(R)$  and  $M$  an  $R$ -module, this translates as follows:  $M$  is in the Auslander class  $\mathcal{A}_C(R)$  if the natural map  $\gamma_M^C : M \rightarrow \mathrm{Hom}_R(C, C \otimes_A M)$  is an isomorphism and  $\mathrm{Tor}_i^R(C, M) = 0 = \mathrm{Ext}_R^i(C, C \otimes_k M)$  for all  $i > 0$ .

**Definition 2.11.** Let  $C \in \mathfrak{S}(A)$  and  $M \in D_b^f(A)$ . Then  $M$  is *derived  $C$ -reflexive* if the natural biduality morphism  $\delta_M^C : M \rightarrow \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(M, C), C)$  is an isomorphism in  $D(A)$  and  $\mathbf{R}\mathrm{Hom}_A(M, C) \in D_b^f(A)$ .

If we are working over  $R$ , where  $C \in \mathfrak{S}_0(R)$  and  $M$  a finitely generated  $R$ -module, this translates to  $\mathrm{G}_C\text{-dim}(M) < \infty$ .

The next result is useful for the proof of Theorem 1.3.

**Lemma 2.12** ([15]). *Assume  $A$  is mildly noetherian and let  $C \in \mathfrak{S}(A)$ . For  $M \in D(A)$ , the following conditions are equivalent.*

- (a)  $M \simeq 0$ ,
- (b)  $C \otimes_A^L M \simeq 0$ , and
- (c)  $\mathbf{R}\mathrm{Hom}_A(C, M) \simeq 0$ .

*When  $M \in D_+^f(A)$ , these are equivalent to the following:*

- (d)  $\mathbf{R}\mathrm{Hom}_A(M, C) \simeq 0$ .

**Notation 2.13.** Let  $B, C$  be DG  $A$ -modules. Then  $B \sim C$ , if there exists an integer  $n$  such that  $B \simeq \Sigma^n C$ .

The next result is a DG version of a result of Araya et al. [2, (5.3)], see [11, Lemma 3.2].

**Lemma 2.14.** *Assume  $A$  is local and  $B, C \in \mathfrak{S}(A)$ . Then  $B \approx C$  if and only if  $B \simeq \Sigma^n C$  for some integer  $n$  in  $D(A)$ .*

**Proof:** One implication is straightforward since  $C \in \mathcal{B}_C(A)$  and  $B \in \mathcal{B}_B(A)$ .

For the other implication, assume  $B \approx C$ . Thus  $B \in \mathcal{B}_C(A)$  and  $C \in \mathcal{B}_B(A)$ . Thus  $B \simeq C \otimes_A^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_A(C, B)$  and  $C \simeq B \otimes_A^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_A(B, C) \simeq C \otimes_A^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_A(C, B) \otimes_A^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_A(B, C)$ . By Apassov [1, Proposition 2], there exists minimal semifree resolutions  $F \xrightarrow{\sim} C$ ,  $G \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_A(C, B)$ , and  $L \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_A(B, C)$  over  $A$ . Since minimal semifree resolutions are unique up to isomorphism, and  $F \simeq C \simeq C \otimes_A^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_A(C, B) \otimes_A^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_A(B, C) \simeq F \otimes_A G \otimes_A L$ , we have  $F \simeq F \otimes_A G \otimes_A L$ . By use of the Poincaré series, it follows that  $\mathbf{R}\mathrm{Hom}_A(C, B) \simeq G \cong \Sigma^n A$  and  $L \cong \Sigma^{-n} A$  for some integer  $n$ . Thus  $B \simeq C \otimes_A \Sigma^n A \simeq \Sigma^n C$ .  $\square$

The remainder of this section focuses on  $k$ -complexes.

**Lemma 2.15.** *Let  $L$  be a  $k$ -complex. Then  $L$  is semiprojective over  $k$ .*

**Proof:** Notice that  $L^\natural$  is free over  $k = k^\natural$ , therefore projective. Now by [4, 3.9.1]  $\partial^k = 0$  implies  $H(L)$  and  $B(L)$  are DG  $k$ -modules. Also we have  $H(L)$  and  $B(L)$  are projective over  $k$ . Thus, by [4, 3.9.7],  $L$  is semiprojective over  $k$ .  $\square$

**Remark 2.16.** Let  $B'$  and  $B''$  be  $k$ -complexes. By Lemma 2.15, the complexes  $B'$  and  $B''$  are semiprojective over  $k$ . Thus  $B' \otimes_k B'' \simeq B' \otimes_k^{\mathbf{L}} B''$ .

**Fact 2.17.** Let  $A, A', B, B'$  be  $k$ -vector spaces, and let  $\alpha : A \rightarrow B$  and  $\alpha' : A' \rightarrow B'$  be  $k$ -module homomorphisms. If  $\alpha$  and  $\alpha'$  are both isomorphisms, then  $\alpha \otimes_k \alpha'$  is an isomorphism. If  $A, A', B, B' \neq 0$ , then the converse holds.

**Fact 2.18.** Let  $X'$  and  $X''$  be  $k$ -complexes. Then the Künneth formula [14, 10.81] implies that there is an isomorphism  $\bigoplus_{p+q=i} H_p(X') \otimes_k H_q(X'') \xrightarrow{\cong} H_i(X' \otimes_k X'')$  given by  $\overline{x'} \otimes_k \overline{x''} \mapsto \overline{x' \otimes_k x''}$ . Moreover, if  $\alpha' : A' \rightarrow B'$  and  $\alpha'' : A'' \rightarrow B''$  are chain maps over  $k$ , then  $\bigoplus_{p+q=i} H_p(\alpha') \otimes_k H_q(\alpha'')$  is identified with  $H_i(\alpha' \otimes_k \alpha'')$  under this isomorphism.

**Lemma 2.19.** *Let  $X', X'', Y', Y''$  be  $k$ -complexes, and let  $\alpha' : X' \rightarrow Y'$  and  $\alpha'' : X'' \rightarrow Y''$  be chain maps over  $k$ . If  $\alpha'$  and  $\alpha''$  are isomorphisms, then  $\alpha' \otimes_k \alpha'' : X' \otimes_k X'' \rightarrow Y' \otimes_k Y''$  is an isomorphism. If  $X', X'', Y', Y'' \neq 0$ , then the converse holds.*

**Proof:** The forward implication is standard. For the reverse implication, assume  $X', X'', Y', Y'' \neq 0$  and  $\alpha' \otimes_k \alpha''$  is an isomorphism. By definition we have  $(\alpha' \otimes_k \alpha'')_i = \bigoplus_{p+q=i} (\alpha'_p \otimes_k \alpha''_q)$ , so  $\alpha'_p \otimes_k \alpha''_q$  is an isomorphism for all  $p, q$ .

It remains to show that  $\alpha'_p$  and  $\alpha''_q$  are isomorphisms for all  $p, q$ . Let  $X'_{p_0} \neq 0$ ,  $X''_{q_0} \neq 0$ ,  $Y'_{p_1} \neq 0$ , and  $Y''_{q_1} \neq 0$ . Suppose  $X'_p = 0$ . Then  $0 = X'_p \otimes_k X''_{q_1} \xrightarrow[\cong]{\alpha'_p \otimes_k \alpha''_{q_1}} Y'_p \otimes_k Y''_{q_1}$ . Therefore,  $Y'_p \otimes_k Y''_{q_1} = 0$ . Since  $Y''_{q_1} \neq 0$ , we have  $Y'_p = 0$ . So  $\alpha'_p$  is an isomorphism. By a similar argument  $Y'_p = 0$  implies  $\alpha'_p$  is an isomorphism. Assume  $X'_p, Y'_p \neq 0$ . The assumption  $X''_{q_0} \neq 0$  implies that  $Y''_{q_0} \neq 0$ . Therefore,  $\alpha'_p \otimes_k \alpha''_{q_0}$  isomorphism such that  $X'_p, Y'_p, X''_{q_0}, Y''_{q_0} \neq 0$ . Thus Lemma 2.17 implies  $\alpha'_p$  and  $\alpha''_{q_0}$  are isomorphisms.

A symmetric argument shows that  $\alpha''_q$  is an isomorphism for all  $q$ .  $\square$

**Lemma 2.20.** *Let  $A', B', A'', B''$  be  $k$ -complexes, and let  $\alpha' : A' \rightarrow B'$  and  $\alpha'' : A'' \rightarrow B''$  be chain maps over  $k$ . If  $\alpha'$  and  $\alpha''$  are quasiisomorphisms, then  $\alpha' \otimes_k \alpha''$  is a quasiisomorphism. If  $A', B', A'', B'' \neq 0$ , then the converse holds.*

**Proof:** This follows from Facts 2.17-2.18 and Lemma 2.19.  $\square$

**Lemma 2.21.** *Let  $M'$  and  $M''$  be  $k$ -complexes. If  $M'$  and  $M''$  are homologically bounded, then  $M' \otimes_k M''$  is homologically bounded. If  $M', M'' \neq 0$ , then the converse holds.*

**Proof:** For the forward implication, set  $t = \sup(M')$ ,  $w = \sup(M'')$ ,  $s = \inf(M')$ , and  $l = \inf(M'')$ .

Case 1: If  $p + q > t + w$ , then  $H_p(M') \otimes_k H_q(M'') = 0$  because  $p + q > t + w$  implies  $p > t$  or  $q > w$ . Therefore, by Fact 2.18,  $H_i(M' \otimes_k M'') = \bigoplus_{p+q=i} (H_p(M') \otimes_k H_q(M'')) = 0$  for  $i > t + w$ .

Case 2: If  $p + q < s + l$ , then  $H_p(M') \otimes_k H_q(M'') = 0$  because  $p + q < s + l$  implies  $p < s$  or  $q < l$ . Therefore,  $H_i(M' \otimes_k M'') = 0$  for  $i < s + l$ .

For the reverse implication suppose  $H_{p_j}(M') \neq 0$  for infinitely many indices  $j$ . Since  $M'' \neq 0$  there exists an integer  $b$  such that  $H_b(M'') \neq 0$ . Now,  $H_{p_j}(M') \otimes_k H_b(M'') \neq 0$  for infinitely many indices  $j$ . However,  $H_{p_j}(M') \otimes_k H_b(M'') \subset H_{p_j+b}(M' \otimes_k M'')$ . Hence, there is an infinite number of  $\sigma = p_j + b$  such that  $H_\sigma(M' \otimes_k M'') \neq 0$  which is a contradiction since  $M' \otimes_k M''$  is homologically bounded. Thus  $H_{p_j}(M') \neq 0$  for only finitely many  $j$ . Hence  $M'$  is homologically bounded. By a similar argument  $M''$  is homologically bounded.  $\square$

### 3. DG TENSOR PRODUCTS

This section consists of tools for use in the proofs of our main theorems.

**Assumption 3.1.** In this section  $A'$  and  $A''$  are DG  $R$ -algebras and  $A := A' \otimes_R A''$ .

**Lemma 3.2.** *Assume that  $R = k$  is a field. Let  $M'$  and  $M''$  be DG  $A'$ - and  $A''$ -modules respectively. If  $M'$  and  $M''$  are degreewise homologically finite over  $A'$  and  $A''$ , respectively, then  $M' \otimes_k M''$  is degreewise homologically finite over  $A' \otimes_k A''$  under any of the following conditions:*

- (1)  $M'$  is homologically bounded,
- (2)  $M''$  is homologically bounded,
- (3)  $M'$  and  $M''$  are homologically bounded below, or
- (4)  $M'$  and  $M''$  are homologically bounded above.

**Proof:** (1) By Fact 2.18, for all  $i$  we have  $H_i(M' \otimes_k M'') \cong \bigoplus_{p+q=i} H_p(M') \otimes_k H_q(M'')$ . Note that this direct sum is finite because  $M' \in D_b(A')$ .

Now,  $H_p(M')$  is finitely generated over  $H_0(A')$  for all  $p$ , and  $H_q(M'')$  is finitely generated over  $H_0(A'')$  for all  $q$ , by our assumption. Therefore,  $H_p(M') \otimes_k H_q(M'')$  is finitely generated over  $H_0(A') \otimes_k H_0(A'')$  for all  $p$  and  $q$ . Hence  $\bigoplus_{p+q=i} H_p(M') \otimes_k H_q(M'')$  is finitely generated for all  $i$ .

The proofs of parts (2)–(4) are similar to proof of part (1). Notice that in each case the assumptions guarantee the direct sum  $\bigoplus_{p+q=i} H_p(M') \otimes_k H_q(M'')$  is finite.  $\square$

The next result gives us some flexibility for understanding how DG  $A'$ - and  $A''$ -modules yield DG  $A$ -modules.

**Lemma 3.3.** *Let  $X'$  and  $X''$  be DG  $A'$ - and  $A''$ -modules respectively. The map  $\alpha_{X''}^{X'} : X' \otimes_R X'' \rightarrow (A \otimes_{A'} X') \otimes_A (A \otimes_{A''} X'')$  given by  $x' \otimes x'' \mapsto (1 \otimes x') \otimes (1 \otimes x'')$  is an isomorphism of DG  $A$ -modules.*

**Proof:** The given map is the composition of the following sequence of isomorphisms.

$$\begin{aligned}
 X' \otimes_R X'' &\cong (X' \otimes_{A'} (A' \otimes_R A'')) \otimes_{A''} X'' \\
 &\cong ((A' \otimes_R A'') \otimes_{A'} X') \otimes_{A''} X'' \\
 &\cong (A \otimes_{A'} X') \otimes_{A''} X'' \\
 &\cong (A \otimes_{A'} X') \otimes_A (A \otimes_{A''} X'')
 \end{aligned}$$

It is straightforward to show that  $\alpha$  is  $A$ -linear.  $\square$

**Lemma 3.4.** *If  $P'$  is a semiprojective DG  $A'$ -module and  $P''$  is semiprojective DG  $A''$ -module, then  $P' \otimes_R P''$  is semiprojective over  $A$ .*

**Proof:** By Lemma 3.3, we have  $P' \otimes_R P'' \cong (A \otimes_{A'} P') \otimes_A (A \otimes_{A''} P'')$  as DG  $A$ -modules.

The fact that  $P'$  is semiprojective over  $A'$  implies that  $A \otimes_{A'} P'$  is semiprojective over  $A$  because

$$\mathrm{Hom}_A(A \otimes_{A'} P', -) \cong \mathrm{Hom}_{A'}(P', \mathrm{Hom}_A(A, -)) \cong \mathrm{Hom}_{A'}(P', -).$$

Similarly,  $A \otimes_{A''} P''$  is semiprojective over  $A$ . Now  $X, Y$  semiprojective over  $A$  implies  $X \otimes_A Y$  is semiprojective over  $A$  because  $\mathrm{Hom}_A(X \otimes_A Y, -) \cong \mathrm{Hom}_A(Y, \mathrm{Hom}_A(X, -))$ . Therefore,  $A \otimes_{A'} P'$  semiprojective over  $A$  and  $A \otimes_{A''} P''$  semiprojective over  $A$  imply that  $(A \otimes_{A'} P') \otimes_A (A \otimes_{A''} P'') \cong P' \otimes_R P''$  is semiprojective.  $\square$

**Lemma 3.5.** *Assume that  $R = k$  is a field. Let  $M'$  and  $M''$  be DG  $A'$ - and  $A''$ -modules respectively. If  $P' \xrightarrow[\cong]{\alpha'} M'$  and  $P'' \xrightarrow[\cong]{\alpha''} M''$  are semiprojective resolutions over  $A'$  and  $A''$ , respectively, then  $P' \otimes_k P'' \xrightarrow[\cong]{\alpha' \otimes_k \alpha''} M' \otimes_k M''$  is a semiprojective resolution over  $A$ .*

**Proof:** Notice  $P' \otimes_k P''$  is semiprojective over  $A$  and  $P' \otimes_k P'' \xrightarrow{\alpha' \otimes_k \alpha''} M' \otimes_k M''$  is a quasiisomorphism by Lemmas 3.4 and 2.20.  $\square$

Our next result is similar in flavor to Lemma ??.

**Lemma 3.6.** *Let  $X'$  and  $X''$  be  $A'$ - and  $A''$ -modules, respectively. The map*

$$\tilde{\gamma}_{Y', Y''}^{X', X''} : (X' \otimes_{A'} Y') \otimes_R (X'' \otimes_{A''} Y'') \rightarrow (X' \otimes_R X'') \otimes_A (Y' \otimes_R Y'')$$

*given by  $(x' \otimes y') \otimes (x'' \otimes y'') \mapsto (-1)^{|y'| |x''|} (x' \otimes x'') \otimes (y' \otimes y'')$  is an isomorphism of DG  $A$ -modules.*

**Proof:** Lemma 3.3 gives the first and last isomorphisms in the following display. The second and third isomorphisms are by associativity, commutativity, etc. of tensor products.

$$\begin{aligned} (X' \otimes_R X'') \otimes_A (Y' \otimes_R Y'') &\cong [(A \otimes_{A'} X') \otimes_A (A \otimes_{A''} X'')] \otimes_A [(A \otimes_{A'} Y') \otimes_A (A \otimes_{A''} Y'')] \\ &\cong (A \otimes_{A'} X') \otimes_A (A \otimes_{A'} Y') \otimes_A (A \otimes_{A''} X'') \otimes_A (A \otimes_{A''} Y'') \\ &\cong (A \otimes_{A'} (X' \otimes_{A'} Y')) \otimes_A (A \otimes_{A''} (X'' \otimes_{A''} Y'')) \\ &\cong (X' \otimes_{A'} Y') \otimes_R (X'' \otimes_{A''} Y'') \end{aligned}$$

It is straightforward to show that  $\tilde{\gamma}_{Y', Y''}^{X', X''}$  is the composition of the displayed isomorphisms and is  $A$ -linear.  $\square$

**Lemma 3.7.** *Assume that  $R = k$  is a field. Then the morphism*

$$\gamma_{Y', Y''}^{X', X''} : (X' \otimes_{A'}^{\mathbf{L}} Y') \otimes_k (X'' \otimes_{A''}^{\mathbf{L}} Y'') \rightarrow (X' \otimes_k X'') \otimes_A^{\mathbf{L}} (Y' \otimes_k Y'')$$

*induced by the morphism  $\tilde{\gamma}_{Q', Q''}^{P', P''}$  from Lemma 3.6 is an isomorphism in  $D(A)$ .*

**Proof:** Let  $P' \xrightarrow{\cong} X'$ ,  $P'' \xrightarrow{\cong} X''$ ,  $Q' \xrightarrow{\cong} Y'$ , and  $Q'' \xrightarrow{\cong} Y''$  be semiprojective resolutions over  $A'$  and  $A''$  as appropriate. By Lemma 3.6, the map

$$\tilde{\gamma}_{Q', Q''}^{P', P''} : (P' \otimes_{A'} Q') \otimes_k (P'' \otimes_{A''} Q'') \rightarrow (P' \otimes_k P'') \otimes_A (Q' \otimes_k Q'')$$

is an isomorphism of DG  $A$ -modules. Therefore,  $\gamma_{Y', Y''}^{X', X''}$  is an isomorphism in  $D(A)$ .  $\square$

The remainder of this section is devoted to understanding  $\mathbf{R}\mathrm{Hom}_A(N, M)$  for DG  $A$ -modules  $M$  and  $N$  constructed as above.

**Definition 3.8.** Let  $N', M'$  and  $N'', M''$  be DG  $A'$ - and  $A''$ -modules, respectively. Consider elements  $f' \in \mathrm{Hom}_{A'}(N', M')$  and  $f'' \in \mathrm{Hom}_{A''}(N'', M'')$ . Let  $f' \boxtimes f'' : N' \otimes_R N'' \rightarrow M' \otimes_R M''$  be given by  $(f' \boxtimes f'')|_{x' \otimes x''}(x' \otimes x'') = (-1)^{|f''||x'|} f'_{|x'|}(x') \otimes f''_{|x''|}(x'')$ .

**Remark 3.9.** With notation as in Definition 3.8, the map  $f' \boxtimes f''$  is well-defined and  $A$ -linear.

**Example 3.10.** Let  $X'$  and  $X''$  be  $R$ -complexes. Then we have  $\partial^{X' \otimes_R X''} = (\partial^{X'} \boxtimes \mathrm{id}) + (\mathrm{id} \boxtimes \partial^{X''})$ .

**Definition 3.11.** Let  $N', M'$  and  $N'', M''$  be DG  $A'$ - and  $A''$ -modules, respectively. Let

$$\tilde{\eta}_{M', M''}^{N', N''} : \mathrm{Hom}_{A'}(N', M') \otimes_R \mathrm{Hom}_{A''}(N'', M'') \rightarrow \mathrm{Hom}_A(N' \otimes_R N'', M' \otimes_R M'')$$

be given by  $f' \otimes f'' \mapsto f' \boxtimes f''$ .

**Remark 3.12.** The map  $\tilde{\eta}_{M', M''}^{N', N''}$  is a well-defined morphism of DG  $A$ -modules.

**Proposition 3.13.** Assume that  $R = k$  is a field. If  $N', N''$  are degreewise finite, semifree, bounded below DG  $A'$ - and  $A''$ -modules, respectively, and  $M', M''$  are bounded above DG  $A'$ - and  $A''$ -modules, respectively, then the morphism  $\tilde{\eta}_{M', M''}^{N', N''}$  is an isomorphism of DG  $A$ -modules.

**Proof:** It suffices to show that the morphism

$$\tilde{\eta}_{M', M''}^{N', N''} : \mathrm{Hom}_{A'}(N', M') \otimes_k \mathrm{Hom}_{A''}(N'', M'') \rightarrow \mathrm{Hom}_A((N' \otimes_k N''), (M' \otimes_k M''))$$

is an isomorphism. Therefore, without loss of generality, assume that all differentials are 0. Thus  $N' \cong \bigoplus_{p \geq p_0} \Sigma^p(A')^{\beta'_p}$  and  $N'' \cong \bigoplus_{q \geq q_0} \Sigma^q(A'')^{\beta''_q}$  for some integers  $\beta'_p, \beta''_q \geq 0$ .

Special case: Assume  $N' = A'$  and  $N'' = A''$ . Set  $\tilde{\eta} = \tilde{\eta}_{M', M''}^{A', A''}$ . It is straightforward to show that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Hom}_{A'}(A', M') \otimes_k \mathrm{Hom}_{A''}(A'', M'') & \xrightarrow{\cong} & M' \otimes_k M'' \\ \tilde{\eta} \downarrow & \nearrow \cong & \\ \mathrm{Hom}_A(A, M' \otimes_k M'') & & \end{array}$$

Hence  $\tilde{\eta}$  is an isomorphism in this case.

General case: Set  $\tilde{\eta}' = \tilde{\eta}_{M', M''}^{N', N''}$ . First we have

$$N' \otimes_k N'' \cong \bigoplus_{p \geq p_0} \bigoplus_{q \geq q_0} \Sigma^{p+q}(A' \otimes_k A'')^{\beta'_p \beta''_q}.$$

Now, for all  $m \in \mathbb{Z}$ , our boundedness condition on  $M'$  implies that

$$\begin{aligned} \mathrm{Hom}_{A'}(N', M')_m &\cong \mathrm{Hom}_{A'} \left( \bigoplus_{p \geq p_0} \Sigma^p(A')^{\beta'_p}, M' \right)_m \\ &\cong \prod_{p \geq p_0} \mathrm{Hom}_{A'}(\Sigma^p(A')^{\beta'_p}, M')_m \\ &= \bigoplus_{p \geq p_0} \mathrm{Hom}_{A'}(\Sigma^p(A')^{\beta'_p}, M')_m. \end{aligned}$$

Similarly, for all  $n \in \mathbb{Z}$ , we have

$$\begin{aligned} \mathrm{Hom}_{A''}(N'', M'')_n &\cong \mathrm{Hom}_{A''} \left( \bigoplus_{q > q_0} \Sigma^q (A'')^{\beta''_q}, M'' \right)_n \\ &\cong \bigoplus_{q \geq q_0} \mathrm{Hom}_{A''} (\Sigma^q (A'')^{\beta''_q}, M'')_n. \end{aligned}$$

The domain of  $\tilde{\eta}'_i$  decomposes as follows.

$$\begin{aligned} &[\mathrm{Hom}_{A'}(N', M') \otimes_k \mathrm{Hom}_{A''}(N'', M'')]_i \\ &\cong \bigoplus_{m+n=i} \left[ \mathrm{Hom}_{A'} \left( \bigoplus_{p \geq p_0} \Sigma^p (A')^{\beta'_p}, M' \right)_m \otimes_k \mathrm{Hom}_{A''} \left( \bigoplus_{q \geq q_0} \Sigma^q (A'')^{\beta''_q}, M'' \right)_n \right] \\ &\cong \bigoplus_{m+n=i} \bigoplus_{p \geq p_0} \bigoplus_{q \geq q_0} \left[ \mathrm{Hom}_{A'} (\Sigma^p A', M)_m^{\beta'_p} \otimes_k \mathrm{Hom}_{A''} (\Sigma^q A'', M'')_n^{\beta''_q} \right] \\ &\cong \bigoplus_{m+n=i} \bigoplus_{p \geq p_0} \bigoplus_{q \geq q_0} \Sigma^{-p-q} [\mathrm{Hom}_{A'} (A', M)_m \otimes_k \mathrm{Hom}_{A''} (A'', M'')_n]^{\beta'_p \beta''_q} \\ &\cong \bigoplus_{p \geq p_0} \bigoplus_{q \geq q_0} \Sigma^{-p-q} \left[ \bigoplus_{m+n=i} \mathrm{Hom}_{A'} (A', M)_m \otimes_k \mathrm{Hom}_{A''} (A'', M'')_n \right]^{\beta'_p \beta''_q} \\ &\cong \bigoplus_{p \geq p_0} \bigoplus_{q \geq q_0} \Sigma^{-p-q} [\mathrm{Hom}_{A'} (A', M) \otimes_k \mathrm{Hom}_{A''} (A'', M'')]_i^{\beta'_p \beta''_q} \end{aligned}$$

Next, we consider the codomain in degree  $i$ .

$$\begin{aligned} \mathrm{Hom}_A(N' \otimes_k N'', M' \otimes_k M'')_i &\cong \mathrm{Hom}_A \left( \left( \bigoplus_{p \geq p_0} \Sigma^p (A')^{\beta'_p} \right) \otimes_k \left( \bigoplus_{q \geq q_0} \Sigma^q (A'')^{\beta''_q} \right), M' \otimes_k M'' \right)_i \\ &\cong \mathrm{Hom}_A \left( \bigoplus_{p \geq p_0} \bigoplus_{q \geq q_0} \Sigma^{p+q} (A' \otimes_k A'')^{\beta'_p \beta''_q}, M' \otimes_k M'' \right)_i \\ &\cong \bigoplus_{p \geq p_0} \bigoplus_{q \geq q_0} \mathrm{Hom}_A (\Sigma^{p+q} (A' \otimes_k A'')^{\beta'_p \beta''_q}, M' \otimes_k M'')_i \\ &\cong \bigoplus_{p \geq p_0} \bigoplus_{q \geq q_0} \Sigma^{-p-q} \mathrm{Hom}_A (A' \otimes_k A'', M' \otimes_k M'')_i^{\beta'_p \beta''_q} \end{aligned}$$

It is straightforward to show that  $\tilde{\eta}$  is compatible with direct sums and shifts. Therefore, we have  $\tilde{\eta}' = \bigoplus_{p \geq p_0} \bigoplus_{q \geq q_0} \Sigma^{-p-q} \tilde{\eta}$ . Since  $\tilde{\eta}$  is an isomorphism by our special case, we conclude that  $\tilde{\eta}'$  is an isomorphism.  $\square$

**Remark 3.14.** Assume that  $R = k$  is a field. Let  $N'$  and  $N''$  be DG  $A'$ - and  $A''$ -modules respectively. Let  $P' \xrightarrow{\sim} N'$  and  $P'' \xrightarrow{\sim} N''$  be semiprojective resolutions over  $A'$  and  $A''$ , respectively. By Lemma 3.5, we have that  $P' \otimes_k P'' \xrightarrow{\sim} N' \otimes_k N''$  is a semiprojective resolution over  $A$ . Therefore,  $\tilde{\eta}_{M', M''}^{P', P''} : \mathrm{Hom}_{A'}(P', M') \otimes_k \mathrm{Hom}_{A''}(P'', M'') \rightarrow \mathrm{Hom}_A(P' \otimes_k P'', M' \otimes_k M'')$  represents a well-defined morphism  $\eta_{M', M''}^{N', N''} : \mathbf{R}\mathrm{Hom}_{A'}(N', M') \otimes_k \mathbf{R}\mathrm{Hom}_{A''}(N'', M'') \rightarrow \mathbf{R}\mathrm{Hom}_A(N' \otimes_k N'', M' \otimes_k M'')$  in  $D(A)$ .



For the next result, notice if  $A'$  and  $A''$  are weakly noetherian, then DG modules  $N' \in D_+^f(A')$  and  $N'' \in D_+^f(A'')$  admit degreewise finite semifree resolutions by Fact 2.8.

**Proposition 3.15.** *Assume that  $R = k$  is a field. Let  $N' \in D_+^f(A')$  and  $N'' \in D_+^f(A'')$  admit degreewise finite semifree resolutions over  $A'$  and  $A''$ , respectively, and  $M' \in D_-(A')$  and  $M'' \in D_-(A'')$ . Then  $\eta_{M', M''}^{N', N''}$  is an isomorphism in  $D(A)$ .*

**Proof:** Notice that  $M'$  and  $M''$  homologically bounded above implies there exists  $L'$  and  $L''$  such that  $\alpha' : M' \xrightarrow{\cong} L'$  and  $\alpha'' : M'' \xrightarrow{\cong} L''$  where  $L'$  and  $L''$  are bounded above. Therefore, we can replace  $M'$  and  $M''$  by  $L'$  and  $L''$  to assume that  $M'$  and  $M''$  are bounded above. By assumption, there exist semifree resolutions  $P' \xrightarrow{\cong} N'$  and  $P'' \xrightarrow{\cong} N''$  such that  $P', P''$  are bounded below and degreewise finite. Therefore, we can replace  $N'$  and  $N''$  by  $P'$  and  $P''$  respectively to assume that  $N'$  and  $N''$  are semifree, bounded below, and degreewise finite. The result now follows from Lemma 3.13.  $\square$

#### 4. SEMIDUALIZING DG MODULES

In this section we prove the main result of this paper and document a few corollaries.

**Assumption 4.1.** In this section  $k$  is a field,  $A'$  and  $A''$  are DG  $k$ -algebras such that  $A' \not\cong 0 \not\cong A''$ , and  $A := A' \otimes_k A''$ .

The next two results are the keys for proving Theorem 1.3 from the introduction.

**Theorem 4.2.** *If  $M'$  and  $M''$  are semidualizing over  $A'$  and  $A''$ , respectively, then  $M' \otimes_k M''$  is semidualizing over  $A$ . If  $A'$  and  $A''$  are mildly noetherian and  $M' \in D_b^f(A')$ ,  $M'' \in D_b^f(A'')$ , then the converse holds.*

**Proof:** Step 1. Note that  $A', A'' \not\cong 0$ , by Assumption 4.1. Thus we have  $A \not\cong 0$ , e.g., by the Künneth formula.

Step 2. If  $M' \in \mathfrak{S}(A')$ , then  $M' \not\cong 0$  because  $\mathbf{R}\mathrm{Hom}_{A'}(M', M') \simeq A' \not\cong 0$ . On the other hand, if  $M' \otimes_k M'' \in \mathfrak{S}(A)$ , then  $M' \otimes_k M'' \not\cong 0$ , so  $M' \not\cong 0$ . Thus, we assume for the remainder of the proof that  $M' \not\cong 0$  and similarly,  $M'' \not\cong 0$ .

Step 3. In the forward implication we assume  $M' \in \mathfrak{S}(A')$  and  $M'' \in \mathfrak{S}(A'')$ , therefore we have  $M' \in D_b^f(A')$  and  $M'' \in D_b^f(A'')$ . Thus, we assume for the remainder of the proof that  $M' \in D_b^f(A')$  and  $M'' \in D_b^f(A'')$ .

Step 4. We assume for the remainder of the proof that  $M'$  and  $M''$  admit degreewise finite semifree resolutions. Notice, in the forward implication, the conditions  $M' \in \mathfrak{S}(A')$  and  $M'' \in \mathfrak{S}(A'')$  guarantee that such resolutions exist; in the reverse implication, since  $A'$  and  $A''$  are weakly noetherian and  $M' \in D_b^f(A')$ ,  $M'' \in D_b^f(A'')$ , Fact 2.8(b) guarantees that such resolutions exist. Note that it follows that the DG module  $M' \otimes_k M'' \in D_b^f(A)$  has such a resolution over  $A$ ; see Lemmas 2.21(a) and 3.2..

Step 5: Consider the following commutative diagram in  $D(A)$ .

$$\begin{array}{ccc} A = A' \otimes_k A'' & \xrightarrow{\chi_{M'}^{A'} \otimes_k \chi_{M''}^{A''}} & \mathbf{R}\mathrm{Hom}_{A'}(M', M') \otimes_k \mathbf{R}\mathrm{Hom}_{A''}(M'', M'') \\ & \searrow \chi_{M' \otimes_k M''}^A & \downarrow \simeq \eta_{N', N''}^{M', M''} \\ & & \mathbf{R}\mathrm{Hom}_A(M' \otimes_k M'', M' \otimes_k M'') \end{array}$$

Notice that the morphism  $\eta_{N', N''}^{M', M''}$  in this diagram is an isomorphism by Lemma 3.15.

In the forward implication, the morphism  $\chi_{M'}^{A'}$  is an isomorphism in  $D(A')$  and  $\chi_{M''}^{A''}$  is an isomorphism in  $D(A'')$ , so  $\chi_{M'}^{A'} \otimes_k \chi_{M''}^{A''}$  is an isomorphism in  $D(A)$  by Lemma 2.20. Therefore, the commutative diagram implies that  $\chi_{M' \otimes_k M''}^A$  is an isomorphism in  $D(A)$ .

In the reverse implication, our commutative diagram with  $\eta_{N', N''}^{M', M''}$  and  $\chi_{M' \otimes_k M''}^A$  isomorphisms in  $D(A)$  imply that  $\chi_{M'}^{A'} \otimes_k \chi_{M''}^{A''}$  is an isomorphism in  $D(A)$ . In particular, we have

$$\mathbf{RHom}_{A'}(M', M') \otimes_k \mathbf{RHom}_{A''}(M'', M'') \simeq A \not\cong 0$$

so  $\mathbf{RHom}_{A'}(M', M'), \mathbf{RHom}_{A''}(M'', M'') \not\cong 0$ . Thus Lemma 2.12 and Lemma 2.20 imply that  $\chi_{M'}^{A'}$  is an isomorphism in  $D(A')$  and  $\chi_{M''}^{A''}$  is an isomorphism in  $D(A'')$ .  $\square$

**Theorem 4.3.** *Fix  $M' \in \mathfrak{S}(A')$  and  $M'' \in \mathfrak{S}(A'')$ , and let  $N' \in D(A')$  and  $N'' \in D(A'')$ . If  $N' \in \mathcal{B}_{M'}(A')$  and  $N'' \in \mathcal{B}_{M''}(A'')$ , then  $N' \otimes_k N'' \in \mathcal{B}_{M' \otimes_k M''}(A)$ . If  $A'$  and  $A''$  are mildly noetherian and  $N', N'' \not\cong 0$ , then the converse holds.*

**Proof:** Step 1: If  $N' \simeq 0$  or  $N'' \simeq 0$ , then  $N' \otimes_k N'' \simeq 0 \in \mathcal{B}_C(A)$ . Therefore, assume for the rest of the proof that  $N', N'' \not\cong 0$ .

Step 2: By Lemma 2.21 we have  $N' \in D_b(A')$  and  $N'' \in D_b(A'')$  if and only if  $N' \otimes_k N'' \in D_b(A)$ . Therefore, assume for the rest of the proof that  $N' \in D_b(A')$  and  $N'' \in D_b(A'')$ .

Step 3: We show that  $\mathbf{RHom}_{A'}(M', N') \in D_b(A')$  and  $\mathbf{RHom}_{A''}(M'', N'') \in D_b(A'')$  if and only if  $\mathbf{RHom}_A(M' \otimes_k M'', N' \otimes_k N'') \in D_b(A)$ . Notice, by Lemma 3.15 we have

$$\mathbf{RHom}_A(M' \otimes_k M'', N' \otimes_k N'') \simeq \mathbf{RHom}_{A'}(M', N') \otimes_k \mathbf{RHom}_{A''}(M'', N'')$$

in  $D(A)$ . Now, by Lemma 2.12, since  $M' \in \mathfrak{S}(A')$  and  $N' \not\cong 0$  we have  $\mathbf{RHom}_{A'}(M', N') \not\cong 0$ . Similarly,  $\mathbf{RHom}_{A''}(M'', N'') \not\cong 0$ . Therefore, by Lemma 2.21 parts (a) and (b) we have  $\mathbf{RHom}_{A'}(M', N') \in D_b(A')$  and  $\mathbf{RHom}_{A''}(M'', N'') \in D_b(A'')$  if and only if  $\mathbf{RHom}_A(M' \otimes_k M'', N' \otimes_k N'') \in D_b(A)$ .

Therefore, assume for the rest of the proof  $\mathbf{RHom}_{A'}(M', N') \in D_b(A')$  and  $\mathbf{RHom}_{A''}(M'', N'') \in D_b(A'')$ .

Step 4: We need to show that  $\xi_{N' \otimes_k N''}^{M' \otimes_k M''}$  is an isomorphism in  $D(A)$  if and only if  $\xi_{N'}^{M'}$  and  $\xi_{N''}^{M''}$  are isomorphisms in  $D(A')$  and  $D(A'')$ , respectively. Consider the following commutative diagram in  $D(A)$ .

$$\begin{array}{ccc} (M' \otimes_A^{\mathbf{L}} \mathbf{RHom}_{A'}(M', N')) \otimes_k (M'' \otimes_A^{\mathbf{L}} \mathbf{RHom}_{A''}(M'', N'')) & & \\ \gamma_{\mathbf{RHom}_{A'}(M', N'), \mathbf{RHom}_{A''}(M'', N'')}^{M', M''} \downarrow \simeq & \searrow \xi_{N'}^{M'} \otimes_k \xi_{N''}^{M''} & \\ (M' \otimes_k M'') \otimes_A^{\mathbf{L}} (\mathbf{RHom}_{A'}(M', N') \otimes_k \mathbf{RHom}_{A''}(M'', N'')) & & N' \otimes_k N'' \\ (M' \otimes_k M'') \otimes_A^{\mathbf{L}} \eta_{N', N''}^{M', M''} \downarrow \simeq & \nearrow \xi_{N' \otimes_k N''}^{M' \otimes_k M''} & \\ (M' \otimes_k M'') \otimes_A^{\mathbf{L}} \mathbf{RHom}_A(M' \otimes_k M'', N' \otimes_k N'') & & \end{array}$$

Notice that  $\gamma_{\mathbf{RHom}_{A'}(M', N'), \mathbf{RHom}_{A''}(M'', N'')}^{M', M''}$  and  $(N' \otimes_k N'') \otimes_A^{\mathbf{L}} \eta_{N', N''}^{M', M''}$  are isomorphisms by Lemmas 3.7 and 3.15. Thus we have that  $\xi_{N' \otimes_k N''}^{M' \otimes_k M''}$  is an isomorphism if and only if  $\xi_{N'}^{M'} \otimes_k \xi_{N''}^{M''}$  is an isomorphism, that is, if and only if  $\xi_{N'}^{M'}$  and  $\xi_{N''}^{M''}$  are isomorphisms by Lemma 2.20 and Lemma 2.12. (Note that this uses the following: by Lemma 2.12, since  $N' \not\cong 0$  and  $M' \in \mathfrak{S}(A')$  we have  $\mathbf{RHom}_{A'}(M', N') \not\cong 0$  and furthermore,  $M' \otimes_A^{\mathbf{L}} \mathbf{RHom}_{A'}(M', N') \not\cong 0$ .)  $\square$

The next two results are proved similarly to Theorem 4.3.

**Theorem 4.4.** Fix  $M' \in \mathfrak{S}(A')$  and  $M'' \in \mathfrak{S}(A'')$  and let  $N' \in D(A')$  and  $N'' \in D(A'')$ . If  $N' \in \mathcal{A}_{M'}(A')$  and  $N'' \in \mathcal{A}_{M''}(A'')$ , then  $N' \otimes_k N'' \in \mathcal{A}_{M' \otimes_k M''}(A)$ . If  $A'$  and  $A''$  are mildly noetherian and  $N' \not\approx 0$  and  $N'' \not\approx 0$ , then the converse holds.

**Theorem 4.5.** Fix  $M' \in \mathfrak{S}(A')$  and  $M'' \in \mathfrak{S}(A'')$  and let  $N' \in D_b^f(A')$  and  $N'' \in D_b^f(A'')$ . If  $N'$  is derived  $M'$  reflexive over  $A'$  and  $N''$  is derived  $M''$  reflexive over  $A''$ , then  $N' \otimes_k N''$  is derived  $M' \otimes_k M''$  reflexive over  $A$ . If  $A'$  and  $A''$  are mildly noetherian and  $N' \not\approx 0$  and  $N'' \not\approx 0$ , then the converse holds.

In the next result, we use the notation of ??.

**Theorem 4.6.** Assume  $M', N' \in \mathfrak{S}(A')$  and  $M'', N'' \in \mathfrak{S}(A'')$ . If  $M' \approx N'$  and  $M'' \approx N''$ , then  $M' \otimes_k M'' \approx N' \otimes_k N''$ . If  $A'$  and  $A''$  are mildly noetherian, then the converse holds.

**Proof:** By a symmetric argument, this is a consequence of Theorem 4.3.  $\square$

**Theorem 4.7.** Assume  $A'$  and  $A''$  are mildly noetherian. Then the map  $\psi : \overline{\mathfrak{S}}(A') \times \overline{\mathfrak{S}}(A'') \rightarrow \overline{\mathfrak{S}}(A)$  given by  $\psi(C', C'') = C' \otimes_k C''$  is well-defined and injective.

**Proof:** This follows from Theorem 4.2 with Theorem 4.6. For instance, assume  $\psi(M', M'') = \psi(N', N'')$ . Then  $M' \otimes_k M'' \approx N' \otimes_k N''$ . Thus,  $M' \approx N'$  and  $M'' \approx N''$  by Theorem 4.6.  $\square$

**4.8** (Proof of Theorem 1.3). (a): This follows from Theorem 4.2.

(b): The map  $\psi$  being well-defined is due to part (a). The map  $\psi$  being injective is a special case of Theorem 4.7 due to Lemma 2.14.  $\square$

We conclude by documenting some special cases of the above results.

**Corollary 4.9.** Let  $R_i$  be a local  $k$ -algebra for  $i = 1, 2$ . Let  $X_i$  be a finitely generated  $R_i$ -module for  $i = 1, 2$ .

- (1) One has  $X_1 \otimes_k X_2 \in \mathfrak{S}_0(R_1 \otimes_k R_2)$  if and only if  $X_i \in \mathfrak{S}_0(R_i)$  for  $i = 1, 2$ .
- (2) The map  $\psi : \mathfrak{S}_0(R_1) \times \mathfrak{S}_0(R_2) \rightarrow \mathfrak{S}_0(R_1 \otimes_k R_2)$  given by  $\psi(C_1, C_2) = C_1 \otimes_k C_2$  is well-defined and injective.

**Corollary 4.10.** Let  $R_i$  be a local  $k$ -algebra for  $i = 1, 2$ . Let  $X_i \in D_b^f(R_i)$  for  $i = 1, 2$ .

- (1) One has  $X_1 \otimes_k X_2 \in \mathfrak{S}(R_1 \otimes_k R_2)$  if and only if  $X_i \in \mathfrak{S}(R_i)$  for  $i = 1, 2$ .
- (2) The map  $\psi : \mathfrak{S}(R_1) \times \mathfrak{S}(R_2) \rightarrow \mathfrak{S}(R_1 \otimes_k R_2)$  given by  $\psi(C_1, C_2) = C_1 \otimes_k C_2$  is well-defined and injective.

**Corollary 4.11.** Let  $R_i$  be a  $k$ -algebra for  $i = 1, 2$ . Let  $X_i \in D_b^f(R_i)$  for  $i = 1, 2$ . Then the map  $\psi : \overline{\mathfrak{S}}(R_1) \times \overline{\mathfrak{S}}(R_2) \rightarrow \overline{\mathfrak{S}}(R_1 \otimes_k R_2)$  given by  $\psi(C_1, C_2) = C_1 \otimes_k C_2$  is well-defined and injective.

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